JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **214,** 219–232 (1997)
ARTICLE NO. AY975613

Periodic Solutions of a Second Order Nonlinear System

Nguyen Phuong Các*

Department of Mathematics, University of Iowa, Iowa City, Iowa 52242

Submitted by Ernst Adams

We discuss the existence of periodic solutions of the system

$$x''(t) + \frac{d}{dt}[\operatorname{grad} F(x(t))] + g(t, x(t)) = e(t)$$

under various asymptotic behaviours of g.

1. INTRODUCTION

Throughout the paper, $F : \mathbb{R}^N \to \mathbb{R}$ $(N \geq 1)$ is twice continuously differentiable; g: $\mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is such that $g(t, s) = g(t + 2\pi, s)$ for almost all (a.a.) $t \in \mathbb{R}$, all $s \in \mathbb{R}^N$, $g(t, \cdot)$ is continuous for a.a. $t \in \mathbb{R}$, $g(\cdot, s)$ is measurable for each $s \in \mathbb{R}^N$, and for each $\rho > 0$ there exists a 2π-periodic function $a_\rho : \mathbb{R} \to \mathbb{R}$ whose restriction to $[0, 2\pi]$ belongs to $L^1[0, 2\pi]$ (we shall say that a_ρ is in L^1) such that

$$|g(t, s)| \leq a_\rho(t) \qquad \text{a.a. } t \in \mathbb{R}, \forall s \in \mathbb{R}^N, |s| \leq \rho, \tag{1}$$

where $|s|$ is the Euclidean norm of $s \in \mathbb{R}^N$ (we shall say that $g(\cdot, \cdot)$ is of L^1-Carathéodory type); and $e : \mathbb{R} \to \mathbb{R}^N$ is 2π-periodic and is in L^1. Additional conditions on the above functions will be imposed in specific situations later. We are concerned with the existence of a 2π-periodic solution (periodic solution in short) of the system

$$x''(t) + \frac{d}{dt}[\operatorname{grad} F(x(t))] + g(t, x(t)) = e(t). \tag{2}$$

* E-mail address: cac@math.uiowa.edu.

0022-247X/97 $25.00

This problem has been extensively studied, mostly when $N = 1$. Then one condition often imposed is

$$\exists d > 0 \text{ such that } sg(t, s) \geq 0, \qquad \text{a.a. } t, \forall |s| \geq d. \tag{3}$$

The beginning of this study of Liénard's equation can perhaps be attributed to A. C. Lazer's works [9, 10] which treat the case $F(s) = c|s|^2$ where c is a constant and g is independent of t. Under the same assumption (3) together with some restriction on the growth of g when $s \to \pm\infty$, the work of Lazer has been generalized in many directions; cf., e.g., [13, 17, 14–16, 18] and their reference lists. The condition dual to (3),

$$\exists d > 0 \text{ s.t. } sg(t, s) \leq 0, \qquad \text{a.a. } t, \forall |s| \geq d, \tag{4}$$

has also been considered; cf. e.g., [13, 17] where g is independent of t. Many other works deal with conditions different from (3) and (4), e.g., involving F; cf., e.g., [1, 2]. The case $N > 1$ has been studied far less often; we mention, e.g., [9, 13] where a modification of condition (3) is introduced. No proof is indicated for $N > 1$ in [9]. It might be appropriate to mention here that, as it seems to us, a significant difficulty in attempting to adapt methods in the literature devised for the case $N = 1$ to the case $N > 1$ is that if, for the sake of simplicity, we assume that g is independent of t, then for a continuous function $x: \mathbb{R}^N \to \mathbb{R}^N$, $g(x(t)) \neq 0 \;\; \forall t$ is incompatible with $\int_0^{2\pi} g(x(t))\, dt = 0$ when $N = 1$ but when $N > 1$ the above incompatibility no longer holds in general as can be seen by simple examples.

In this paper we shall show that when $N > 1$, (4) can be strengthened slightly to guarantee the existence of a periodic solution of (2). Modifying condition (3) for $N > 1$ seems more difficult. Without the strengthened version of (4), we only succeed when the equation has appropriate symmetries; cf. Theorems 3 and 4 below.

To conclude, we mention that results involving the periodic Fučik spectrum instead of the eigenvalues as in the papers already cited have also been obtained; cf., e.g., [8] and its reference list. Furthermore, if $F(s) = 0$ and $g(t, s) = \operatorname{grad}_s G(t, s)$ where $G(t, s): \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ then variational methods can be applied and results involving the asymptotic behavior of $G(t, s)$ are obtained; cf. [4, 7, 6, 5, 11, 3], and their reference lists.

2. NOTATIONS AND TERMINOLOGY

$\langle \cdot, \cdot \rangle$ and $|\cdot|$ are respectively the inner product and the Euclidean norm in $\mathbb{R}^N$. If $\gamma: \mathbb{R} \to \mathbb{R}^N$ is 2π-periodic and its restriction to $[0, 2\pi]$

belongs to $L^p([0, 2\pi])$, $p \geq 1$, then we say that γ is in L^p and let

$$\bar{\gamma} := \frac{1}{2\pi} \int_0^{2\pi} \gamma(t)\, dt, \qquad |\gamma|_p := \left(\int_0^{2\pi} |\gamma(t)|^p\, dt \right)^{1/p}.$$

If γ is bounded, then

$$|\gamma|_\infty = \sup\{t \in \mathbb{R} \,|\, |\gamma(t)|\}.$$

3. THE RESULTS

We shall need the following result established by Mawhin [12, Theorem 4; 13, Lemma 2].

LEMMA 1. *Suppose that* $q : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, $(t, (\xi, \zeta, \lambda)) \to q(t, \xi, \zeta; \lambda)$ *with* $t \in \mathbb{R}$, $\xi, \zeta \in \mathbb{R}^N$, $\lambda \in \mathbb{R}$, *is* 2π*-periodic in the first variable and* L^1*-Carathéodory. Suppose also that there exists a constant c independent of* $\lambda \in (0, 1]$ *such that for every possible* 2π*-periodic solution* $x : \mathbb{R} \to \mathbb{R}^N$ *of the system*

$$x'' = \lambda q(t, x, x'; \lambda), \qquad \lambda \in (0, 1], \tag{5}$$

we have

$$|x|_\infty + |x'|_\infty < c,$$

and that for some $r > c$ *the Brouwer degree* $d(Q_0, B_r, 0)$ *exists and is not zero where* $B_r = \{a \in \mathbb{R}^N \,|\, |a| < r\}$, $\bar{B}_r$ *is the closure of* B_r, *and*

$$Q_0 : \bar{B}_r \to \mathbb{R}^N, \qquad Q_0(a) = \frac{1}{2\pi} \int_0^{2\pi} q(t, a, 0; 0)\, dt. \tag{6}$$

Then the system

$$x'' = q(t, x, x'; 1) \tag{7}$$

has a 2π*-periodic solution.*

Note. This result is proved in [12] when q is continuous in all variables. It is not difficult to see that it remains valid under the conditions given above.

We now prove a kind of universal estimate for periodic solutions valid for all $N \geq 1$ under a very mild restriction on g.

LEMMA 2. *Suppose that there exists a constant* $d > 0$ *such that*

$$\langle g(t,s), s\rangle \le 0 \qquad \text{a.a. } t \in \mathbb{R}, \forall |s| \ge d. \tag{8}$$

Consider the family of Liénard systems

$$x'' + \lambda \frac{d}{dt}[\operatorname{grad} F(x)] + \lambda g(t,x) = \lambda e(t), \qquad 0 < \lambda \le 1, N \ge 1. \tag{9}$$

Then given any $\varepsilon > 0$ *there exists a positive constant* $K(\varepsilon, |a_d|_1, |e|_1)$ *independent of* $\lambda \in (0,1]$ *such that*

$$|x'|_2 \le \varepsilon|\bar{x}| + K(\varepsilon, |a_d|_1, |e|_1), \tag{10}$$

for any solution $x(\cdot)$ *of* (9), *where, it might be recalled,* $|a_d|_1$ *is the* L^1*-norm of the function* a_d *involved in* (1) *and* $\bar{x}$ *is the average value of* $x(\cdot)$ *on* $[0, 2\pi]$.

Note. An estimate similar to (10) is obtained in [13] when g is independent of t, $\lim_{|s|\to\infty}(|g(s)|/|s|) = 0$ instead of (8), and $e(\cdot) \in L^2$.

Proof. We define a function $\gamma : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ as

$$\begin{cases} \gamma(t,s) = g(t,s) & \text{if } |s| \ge d \\ \gamma(t,s) = \dfrac{|s|}{d} g\left(t, \dfrac{d}{|s|}s\right) & \text{if } 0 < |s| < d \\ \gamma(t,0) = 0 & \forall t \in \mathbb{R} \end{cases} \tag{11}$$

and let

$$h(t,s) = g(t,s) - \gamma(t,s). \tag{12}$$

Then it is not hard to verify that

$$\langle \gamma(t,s), s\rangle \le 0, |h(t,s)| \le 2a_d(t), \qquad \text{a.a. } t \in \mathbb{R}, \forall s \in \mathbb{R}^N. \tag{13}$$

In (9) replacing g by $\gamma + h$, taking the inner product with $x(t)$, and integrating over $[0, 2\pi]$ we obtain, because of the periodicity of $x(t)$,

$$\begin{aligned} \int_0^{2\pi} |x'(t)|^2\, dt - \lambda \int_0^{2\pi} \langle \gamma(t,x), x\rangle\, dt - \lambda \int_0^{2\pi} \langle h(t,x), x\rangle\, dt \\ = -\lambda \int_0^{2\pi} \langle e(t), x\rangle\, dt. \end{aligned} \tag{14}$$

Taking into account (13) we have

$$\int_0^{2\pi} |x'(t)|^2 \, dt \le \lambda \int_0^{2\pi} \langle h(t,x) - e(t), \bar{x} + \tilde{x} \rangle \, dt$$

$$\le \int_0^{2\pi} \big(2|a_d(t)| + |e(t)|\big)\big(|\bar{x}| + |\tilde{x}(t)|\big) \, dt, \qquad (15)$$

where $x(\cdot) = \bar{x} + \tilde{x}(\cdot)$. Now

$$\int_0^{2\pi} \big(2|a_d(t)| + |e(t)|\big)|\tilde{x}(t)| \, dt \le |\tilde{x}|_\infty (2|a_d|_1 + |e|_1)$$

$$\le \frac{3}{\pi}|\tilde{x}|_\infty^2 + \frac{\pi}{12}(2|a_d|_1 + |e|_1)^2$$

$$\le \frac{1}{2}|x'|_2^2 + \frac{\pi}{12}(2|a_d|_1 + |e|_1)^2 \qquad (16)$$

in view of Sobolev's inequality

$$|\tilde{x}|_\infty^2 \le \frac{\pi}{6}|\tilde{x}'|_2^2 = \frac{\pi}{6}|x'|_2^2.$$

On the other hand

$$\int_0^{2\pi} \big(2|a_d(t)| + |e(t)|\big)|\bar{x}| \, dt = |\bar{x}|(2|a_d|_1 + |e|_1)$$

$$\le \frac{\varepsilon^2}{2}|\bar{x}|^2 + \frac{1}{2\varepsilon^2}(2|a_d|_1 + |e|_1)^2. \qquad (17)$$

From (15), (16), and (17) we obtain

$$|x'|_2 \le \varepsilon|\bar{x}| + K(\varepsilon, |a_d|_1, |e|_1),$$

where

$$K(\varepsilon, |a_d|_1, |e|_1) = \left(\frac{\pi}{6} + \frac{1}{\varepsilon^2}\right)^{1/2} (2|a_d|_1 + |e|_1). \quad \blacksquare$$

We now sketch the proof of the existence of a periodic solution of (2) when $N = 1$ and condition (4) is satisfied. This result has been proved in [17, Theorem 3] where the *a priori* estimates used are derived by using the Green function of $x'' - \mu^2 x$ ($\mu > 0$) with periodic boundary conditions. It is also included in [13, Theorem 1] with the additional condition $g(s)/s \to 0$ as $|s| \to \infty$. Our proof uses the estimate obtained in Lemma 2 above, i.e., uses neither Green's function nor $g(s)/s \to 0$ as $|s| \to \infty$. Nevertheless our main purpose is to exhibit by means of the proof the essential difficulty, as appeared to us, encountered in attempting to adapt methods devised for scalar equations to systems.

THEOREM 1. *Consider Eq.* (2) *for* $N = 1$. *Suppose that* $\bar{e} = (1/2\pi)\int_0^{2\pi} e(t)\,dt = 0$ *and condition* (4) *is satisfied*: $\exists d > 0$ *s.t.* $g(t,s)s \leq 0$ *a.a.* $t \in \mathbb{R}$, $\forall |s| \geq d$. *Then Eq.* (2) *has a* 2π-*periodic solution*.

Proof. From (9), with $f(s) = (d^2/ds^2)F(s)$, $s \in \mathbb{R}$,

$$x'' + \lambda f(x)x' + \lambda g(t,x) = \lambda e(t), \qquad 0 < \lambda \leq 1,$$

we obtain for all periodic solutions

$$\int_0^{2\pi} g(t, x(t))\,dt = 0. \tag{18}$$

It suffices to prove the theorem assuming the following stronger condition than (4),

$$\exists d > 0 \text{ s.t. } g(t,s)s < 0 \qquad \text{a.a. } t \in \mathbb{R}, \forall |s| \geq d \tag{19}$$

because we can then use a perturbation argument to show that the theorem holds true under condition (4); we refer the reader to [13, Remark 1, p. 26] for details. We claim that for every periodic solution $x(\cdot)$ of (9) there is $\tau \in \mathbb{R}$ such that

$$|x(\tau)| < d. \tag{20}$$

In fact, if $|x(t)| \geq d$ $\forall t \in \mathbb{R}$, then either $x(t) \geq d$ and $g(t, x(t)) < 0$ a.a. $t \in \mathbb{R}$ or $x(t) \leq -d$ and $g(t, x(t)) > 0$ a.a. $t \in \mathbb{R}$ contradicting (18). By Sobolev's inequality we obtain from (10) with $\varepsilon = (1/2)(6/\pi)^{1/2}$

$$|\tilde{x}|_\infty \leq \left(\frac{\pi}{6}\right)^{1/2} |\tilde{x}'|_2 \leq \frac{1}{2}|\bar{x}| + c_1 \tag{21}$$

for every solution $x(\cdot)$ of (9) where $c_1 > 0$ is a constant independent of $\lambda \in (0,1]$. Then because $x(t) = \bar{x} + \tilde{x}(t)$

$$|x(t)| \geq |\bar{x}| - |\tilde{x}|_\infty \geq \frac{1}{2}|\bar{x}| - c_1 \qquad \forall t \in \mathbb{R}. \tag{22}$$

From (20) and (22) we obtain

$$|\bar{x}| \leq 2(c_1 + d) \qquad \forall \lambda \in [0,1]. \tag{23}$$

Using (23) and (21) we then obtain bounds independent of $\lambda \in (0,1]$ for $|x|_\infty$ and $|x'|_\infty$ for every periodic solution of (9). Please see the proof of Theorem 2 below for details. Condition (19) is used to show that $d(Q_0, B_r, 0) = 1$ where Q_0 is defined by (6), B_r is the open interval $-r < s < r$ of $\mathbb{R}$, $r > d$ arbitrary and

$$q(t, \xi, \zeta; \lambda) = -f(\xi)\zeta - g(t, \xi) + e(t), \qquad t, \xi, \zeta \in \mathbb{R}, 0 < \lambda \leq 1.$$

The claim then follows from Lemma 1. ∎

From the above proof we see that estimate (20) is crucial in obtaining, via (23), bounds for solutions of (9). Estimate (20) follows from (18) and (19) when $N = 1$ but when $N > 1$ that does not seem true any longer: Let for $\rho \in \mathbb{R}$, $g(t,(s_1, s_2)) = (-1/(1 + |s|))(s_1, s_2)$ $\forall t$, $s_1, s_2 \in \mathbb{R}$, $x(t) = \rho(\cos t, \sin t)$. For the components g_i of g, $s_i g_i(s) < 0$, $\forall |s_i| > 0$, $i = 1, 2$. $\int_0^{2\pi} g(x(t))\, dt = 0$, yet $|x(t)| \equiv |\rho|$ can be as large as desired. In papers dealing with the case $N = 1$, estimate (20), obtained from an equation similar to (18) in the manner described in the proof above, is used to obtain necessary estimates for solutions of (9) whether condition (19) or its dual where $g(t,s)s > 0$ a.a. t, $\forall |s| > d$ is assumed; cf. [13, Proof of Theorem 1; 15, Proof of Theorem 1; 17, Proofs of Theorems 1 and 3]. Thus methods devised for the case $N = 1$ in those papers do not seem to work when $N > 1$.

Without relying on (20), we shall show that when $N > 1$, condition (4) can be strengthened slightly to guarantee a periodic solution of the system (2).

THEOREM 2. *Consider the system* (2) *for* $N > 1$. *Suppose that* $\bar{e} = \int_0^{2\pi} e(t)\, dt = 0$, (8) *of Lemma* 2 *is satisfied, and*

$$b(t) := \limsup_{|s| \to \infty} \frac{\langle g(t,s), s \rangle}{|s|} \qquad \textit{uniformly a.a. } t \in \mathbb{R}. \tag{24}$$

Suppose also that $b(\cdot) \in L^1$, $b(t) \leq 0$ a.a. t, *and the set*

$$\{0 \leq t \leq 2\pi | b(t) < 0\}$$

has positive measure. Then the system (2) *has a* 2π*-periodic solution.*

Proof. It is not difficult to see that (22) in the proof of Theorem 1 is valid for all $N \geq 1$: $\exists$ contain $c_1 > 0$ such that for every periodic solution $x(\cdot)$ of (9) we have (*from here on* c_i, $i = 1, 2, \ldots,$ *denotes a positive constant* *independent of* $\lambda \in (0,1]$ *and not always the same*)

$$|x(t)| \geq \frac{1}{2}|\bar{x}| - c_1 \qquad \forall t \in \mathbb{R}, 0 < \lambda \leq 1. \tag{25}$$

We now claim that $\exists$ constant $c_2 > 0$ such that for every periodic solution $x(\cdot)$ of (9)

$$|\bar{x}| \leq c_2, \qquad 0 < \lambda \leq 1. \tag{26}$$

Suppose by contradiction that (26) is false, i.e., that there exists a sequence $\{\lambda_n\}_{n=1}^{n=\infty}$ in $(0,1]$ and 2π-periodic solution $x_n(\cdot)$ of (9) with $\lambda = \lambda_n$ such that $|\bar{x}_n| \to \infty$ as $n \to \infty$. Then from (25) we have

$$|x_n(t)| \to \infty \qquad \text{as } n \to \infty, \text{ uniformly in } t. \tag{27}$$

Because $\int_0^{2\pi} e(t)\, dt = 0$, by taking the product of (9)

$$x'' + \lambda \frac{d}{dt}[\operatorname{grad} F(x)] + \lambda g(t,x) = \lambda e(t), \qquad 0 < \lambda \leq 1,$$

with $x(t) = \bar{x} + \tilde{x}(t)$ in $\mathbb{R}^N$ and integrating over $[0, 2\pi]$ we obtain for each $n = 1, 2, \ldots$

$$0 \leq \int_0^{2\pi} \langle g(t, x_n(t)), x_n(t)\rangle\, dt - \int_0^{2\pi} \langle e(t), \tilde{x}_n(t)\rangle\, dt. \tag{28}$$

Because of (24) and (27), given $\varepsilon > 0$ we can find a positive integer $n_0(\varepsilon)$ s.t.

$$n \geq n_0(\varepsilon) \Rightarrow \langle g(t, x_n(t)), x_n(t)\rangle \leq [b(t) + \varepsilon]|x_n(t)|, \qquad \text{a.a. } t \in \mathbb{R}. \tag{29}$$

It follows from (28) and (29) that for all large n

$$0 \leq \left[\min_{0 \leq t \leq 2\pi} |x_n(t)|\right] \int_0^{2\pi} b(t)\, dt + 2\pi\varepsilon |x_n(\cdot)|_\infty + |e(\cdot)|_1 |\tilde{x}_n(\cdot)|_\infty. \tag{30}$$

From (25) we have

$$\left[\min_{0 \leq t \leq 2\pi} |x_n(t)|\right] \int_0^{2\pi} b(t)\, dt \leq \frac{1}{2}|\bar{x}_n| \int_0^{2\pi} b(t)\, dt + c_3. \tag{31}$$

On the other hand from (10) and Sobolev's inequality we obtain

$$|\tilde{x}_n|_\infty \leq \left(\frac{\pi}{6}\right)^{1/2} |\tilde{x}_n'|_2 \leq \left(\frac{\pi}{6}\right)^{1/2} \varepsilon |\bar{x}_n| + c_4. \tag{32}$$

and hence

$$|x_n|_\infty \le \left[1 + \left(\frac{\pi}{6}\right)^{1/2} \varepsilon\right]|\bar{x}_n| + c_4. \tag{33}$$

From (30), (31), (32), and (33) we deduce that for all large n

$$\begin{aligned} 0 \le \frac{1}{2}|\bar{x}_n|\int_0^{2\pi} b(t)\,dt \\ + \left\{2\pi\varepsilon\left[1 + \left(\frac{\pi}{6}\right)^{1/2}\varepsilon\right] + |e|_1\left(\frac{\pi}{6}\right)^{1/2}\varepsilon\right\}|\bar{x}_n| + c_5. \end{aligned} \tag{34}$$

Since $|\bar{x}_n| \to \infty$ as $n \to \infty$ and $\int_0^{2\pi} b(t)\,dt < 0$, by taking $\varepsilon > 0$ sufficiently small in (34) we arrive at a contradiction and (26) is proved.

It follows from (26) and the universal estimate (10) of Lemma 2 that $\exists c_6 > 0$ such that for every solution $x(\cdot)$ of (9)

$$|\tilde{x}'|_2 \le c_6, \qquad 0 < \lambda \le 1. \tag{35}$$

We then have by Sobolev's inequality

$$|\tilde{x}|_\infty \le \left(\frac{\pi}{6}\right)^{1/2} c_6, \qquad 0 < \lambda \le 1. \tag{36}$$

This in conjunction with (26) gives

$$|x|_\infty \le |\bar{x}| + |\tilde{x}|_\infty \le c_7, \qquad 0 < \lambda \le 1 \tag{37}$$

for some constant $c_7 > 0$. Taking the ith component of system (9) we obtain

$$x_i'' + \lambda \sum_{j=1}^{n} \frac{\partial^2 F}{\partial s_j\,\partial s_i}(x(t))x_j' + \lambda g_i(t, x(t)) = \lambda e_i(t), \qquad i = 1, \dots, N. \tag{38}$$

Because $F(\cdot) \in C^2$, from (35), (37), (38), and the fact that g is of L^1-Carathéodory type, i.e., (1) is valid, it follows that

$$|x_i''|_1 \le c_8, \qquad 0 < \lambda \le 1, 1 \le i \le N. \tag{39}$$

Since $x_i(\cdot)$ is a 2π-periodic, scalar function, $\exists \tau_i \in (0, 2\pi)$ such that $x_i'(\tau_i) = 0$. We then deduce from (39) that

$$|x_i'|_\infty \leq 2\pi c_8, \qquad 0 < \lambda \leq 1, 1 \leq i \leq N,$$

i.e.,

$$|x'|_\infty \leq c_9, \qquad 0 < \lambda \leq 1. \tag{40}$$

In view of (37) and (40) the theorem is obtained by applying Lemma 1 provided we can show that

$$d(Q_0, B_r, 0) \neq 0$$

for some ball B_r in $\mathbb{R}^N$ where

$$Q_0(a) = \frac{-1}{2\pi}\int_0^{2\pi} g(t, a)\, dt, \qquad a \in \mathbb{R}^N.$$

In fact, it suffices to put (9) under the form (5) of Lemma 1 with

$$q(t, x, x'; \lambda) \equiv -\frac{d}{dt}[\operatorname{grad} F(x)] - g(t, x) + e(t),$$

for $t \in \mathbb{R}$, $0 < \lambda \leq 1$. Now let

$$\varepsilon_1 = \frac{-\bar{b}}{2} = \frac{-1}{4\pi}\int_0^{2\pi} b(t)\, dt > 0.$$

Because of (24), $\exists r_1 > 0$, s.t.

$$|s| \geq r_1 \Rightarrow \langle g(t, s), s\rangle \leq [b(t) + \varepsilon_1]|s|. \tag{41}$$

Take $r > r_1 + c_7 + c_9$ where c_7 and c_9 come from (37) and (40), respectively. Because of (41) we have for all $a \in \mathbb{R}^N$, $|a| = r$,

$$\langle Q_0(a), a\rangle = \frac{-1}{2\pi}\int_0^{2\pi} \langle g(t, a), a\rangle\, dt \geq -\left[\frac{1}{2\pi}\int_0^{2\pi} b(t)\, dt + \varepsilon_1\right]|a|$$
$$= -\frac{\bar{b}}{2}|a| > 0.$$

Then a simple homotopy gives

$$d(Q_0, B_r, 0) = d(I, B_r, 0) = 1,$$

where I is the identity mapping in $\mathbb{R}^N$. ∎

For $N > 1$, without condition (8) as strengthened by (24) we have not in general succeeded in proving the existence of a periodic solution of the system (2) assuming, e.g., a condition like (3). This we have only managed to do when the system has appropriate symmetries.

THEOREM 3. *Suppose that F, g, and e are all odd*: $F(-s) = -F(s)$, $g(-t, -s) = -g(t, s)$, $e(-t) = -e(t)$, a.a. $t \in \mathbb{R}$, $\forall s \in \mathbb{R}^N$. *Suppose also that there exists a measurable 2π-periodic, odd function $\Gamma : \mathbb{R} \to \mathbb{R}$, $\Gamma(t) \leq 1$* a.a. $t \in \mathbb{R}$ *and the set $\{0 \leq t \leq 2\pi \mid \Gamma(t) < 1\}$ has positive measure such that*

$$\limsup_{|s|\to\infty} \frac{\langle g(t,s), s\rangle}{|s|^2} \leq \Gamma(t) \qquad \textit{uniformly a.a. } t \in \mathbb{R}. \tag{42}$$

Then the system (2) *has a periodic, odd solution.*

Proof. Note that without loss of generality we can assume that $\Gamma(t) \geq \frac{1}{2}$ a.a. $t \in \mathbb{R}$. We seek to apply Theorem 6 of [12]. Accordingly we introduce the homotopic family of systems

$$x'' + \lambda \frac{d}{dt}[\operatorname{grad} F(x)] + \lambda g(t, x) = \lambda e(t), \qquad 0 \leq \lambda \leq 1, \tag{43}$$

and it suffices to show that for all odd, periodic solutions $x(\cdot)$ of (43)

$$|x|_\infty + |x'|_\infty \leq c_{10}, \qquad 0 \leq \lambda \leq 1, \tag{44}$$

for some constant c_{10} independent of λ.

Noting that the average value over $[0, 2\pi]$ of an odd, 2π-periodic function is equal to 0, it can be proved using a familiar compactness argument that under the stated assumptions on $\Gamma(\cdot)$ there exists a constant $\delta > 0$ such that for all absolutely continuous, odd, 2π-periodic functions $x(\cdot)$ whose derivative $x'(\cdot)$ belongs to L^2 we have

$$\int_0^{2\pi} \left\{ |x'(t)|^2 - \Gamma(t)|x(t)|^2 \right\} dt \geq \delta |x'|_2^2 \tag{45}$$

(cf., e.g., [15, Lemma 1]).

By (42) we can choose $\rho > 0$ sufficiently large so that

$$|s| \geq \rho \Rightarrow \langle g(t,s), s\rangle \leq \left(\Gamma(t) + \frac{\delta}{2}\right)|s|^2 \qquad \text{a.a. } t \in \mathbb{R}. \tag{46}$$

Then for the function $\gamma(\cdot, \cdot)$ defined in (11) with ρ replacing d it is not difficult to verify using (46) that

$$\langle \gamma(t,s), s\rangle \leq \left(\Gamma(t) + \frac{\delta}{2}\right)|s|^2 \qquad \text{a.a. } t \in \mathbb{R}, \forall s \in \mathbb{R}^N. \tag{47}$$

With $h(t,s) = g(t,s) - \gamma(t,s)$ we have

$$|h(t,s)| \le 2a_\rho(t) \qquad \text{a.a. } t \in \mathbb{R}, \tag{48}$$

where $a_\rho(\cdot) \in L^1$ is the function involved in the Carathéodory property of $g(\cdot,\cdot)$ in (1). Taking the inner product of (43) with $x(t)$ and integrating over $[0, 2\pi]$ give

$$\int_0^{2\pi} \left\{ |x'(t)|^2 - \lambda \Gamma(t) |x(t)|^2 \right\} dt$$
$$= \lambda \int_0^{2\pi} \left\{ -\Gamma(t) |x(t)|^2 + \langle \gamma(t, x(t)), x(t) \rangle \right\} dt$$
$$+ \lambda \int_0^{2\pi} \langle h(t, x(t)) - e(t), x(t) \rangle \, dt.$$

From (45), (47), (48), and Wirtinger's inequality we then have for any odd periodic solution $x(\cdot)$ of (43) and $0 \le \lambda \le 1$,

$$\delta |x'|_2^2 \le \frac{\delta}{2} |x'|_2^2 + \left(2|a_\rho|_1 + |e|_1\right) |x|_\infty.$$

Using the Sobolev inequality we deduce

$$|x'|_2 \le \frac{2}{\delta} \left(\frac{\pi}{6} \right)^{1/2} \left(2|a_\rho|_1 + |e|_1\right) \tag{49}$$

and

$$|x|_\infty \le \frac{\pi}{3\delta} \left(2|a_\rho|_1 + |e|_1\right). \tag{50}$$

For any component $x_i(\cdot)$, $i = 1, \dots, N$, of $x(\cdot)$ we have

$$x_i''(t) + \lambda \sum_{j=1}^{N} \frac{\partial^2 F}{\partial s_j \, \partial s_i} (x) x_j' + \lambda g_i(t,x) = \lambda e_j(t).$$

From (49) and (50) we deduce

$$|x_i''|_1 \le c_{11}, \qquad i = 1, 2, \dots, N \tag{51}$$

for a constant c_{11} independent of $\lambda \in [0,1]$ for any odd periodic solution $x(\cdot)$ of (43). Because $x_i(\cdot)$ is periodic, for each $i = 1, \dots, N$,

$$\exists \tau_i \in [0, 2\pi] \qquad \text{s.t. } x_i'(\tau_i) = 0. \tag{52}$$

From (51) and (52) we obtain bounds for $|x_i'|_\infty$ independent of $\lambda \in [0,1]$ for each $i = 1, 2, \dots, N$. Hence

$$|x'|_\infty < c_{12}, \qquad 0 \le \lambda \le 1 \tag{53}$$

for some constant c_{12}. Thus (44) is established and the theorem is proved. ∎

Similar to Theorem 3 we have

THEOREM 4. *Suppose that* F, g, *and* e *are such that* $F(-s) = F(s)$, $g(t+\pi, -s) = -g(t,s)$, $e(t+\pi) = -e(t)$, a.a. $t \in \mathbb{R}$, $s \in \mathbb{R}^N$. *Suppose also that there exists a measurable* 2π*-periodic function* $\Gamma : \mathbb{R} \to \mathbb{R}$, $\Gamma(t+\pi) = -\Gamma(t)$, $\Gamma(t) \le 1$ a.a. $t \in \mathbb{R}$ *and*

$$\operatorname{meas}\{0 \le t \le 2\pi | \Gamma(t) < 1\} > 0$$

such that

$$\limsup_{|s| \to \infty} \frac{\langle g(t,s), s\rangle}{|s|^2} \le \Gamma(t) \qquad \textit{uniformly } \text{a.a. } t \in \mathbb{R}.$$

Then the system (2) *has a* 2π*-periodic solution* $x(\cdot)$ *such that*

$$x(t+\pi) = -x(t) \qquad \forall t \in \mathbb{R}.$$

Proof. The proof is the same as for Theorem 3, using Theorem 5 of [12]. ∎

REFERENCES

1. T. A. Burton, On the equation $x'' + f(x)h(x')x' + g(x) = e(t)$, *Ann. Mat. Pura Appl. (4)* **85** (1970), 277–285.
2. T. A. Burton and C. G. Townsend, On the generalized Liénard equation with forcing function, *J. Differential Equations* **4** (1968), 620–633.
3. D. Costa and M. Cuesta, Existence results for resonant perturbations of the Fučik spectrum, preprint.
4. M. Cuesta and J. P. Gossez, A variational approach to nonresonance with respect to the Fučik spectrum, *Nonlinear Anal.* **19** (1992), 487–500.
5. D. G. de Figueiredo and B. Ruf, On the periodic Fučik spectrum and a superlinear Sturm–Liouville equation, *Proc. Roy. Soc. Edinburgh Sect. A* **123** (1993), 95–107.
6. A. Fonda, On the existence of periodic solutions for scalar second order differential equations when only the asymptotic behaviour of the potential is known, *Proc. Amer. Math. Soc.* **119** (1993), 439–445.
7. A. Fonda and A. C. Lazer, Subharmonic solutions of conservative systems with nonconvex potentials, *Proc. Amer. Math. Soc.* **115** (1992), 183–190.

8. J. P. Gossez and P. Omari, Periodic solutions of a second order ordinary differential equation: A necessary and sufficient condition for nonresonance, *J. Differential Equations* **94** (1991), 67–82.
9. A. C. Lazer, On Schauder's fixed point theorem and forced second-order nonlinear oscillations, *in* "Proceedings United States-Japan Seminar on Differential and Functional Equations," pp. 473–478, Benjamin, New York/Amsterdam, 1967.
10. A. C. Lazer, On Schauder's fixed point theorem and forced second-order nonlinear oscillations, *J. Math. Anal. Appl.* **21** (1968), 421–425.
11. M. Cuesta Leon, A nonresonance result with respect to the Fučik spectrum for a second order differential equation, *Differential Integral Equatons* **9** (1996), 11–26.
12. J. Mawhin, Equations intégrales et solutions périodiques des systèmes différentiels non linéaires, *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **55** (1969), 934–947.
13. J. Mawhin, An extension of a theorem of A. C. Lazer on forced non-linear oscillations, *J. Math. Anal. Appl.* **40** (1972), 20–29.
14. J. Mawhin and J. R. Ward, Nonuniform nonresonance conditions at the first two eigenvalues for periodic solutions of forced Liénard and Duffing equations, *Rocky Mountain J. Math.* **12** (1982),643–654.
15. J. Mawhin and J. R. Ward, Periodic solutions of some forced Liénard differential equations at resonance, *Arch. Math. (Basel)* **41** (1983), 337–351.
16. P. Omari, G. Villari, and F. Zanolin, Periodic solutions of the Liénard equation with one-sided growth restrictions, *J. Differential Equations* **67** (1987), 278–293.
17. R. Reissig, Extension of some results concerning the generalized Liénard equation, *Ann. Mat. Pura Appl. (4)* **104** (1975), 269–281.
18. K. Schmitt, Periodic solutions of a forced nonlinear oscillator involving a onesided restoring force, *Arch. Math. (Basel)* **31** (1978), 70–73.